Fractional Dynamical Behavior in Quantum Brownian Motion

Kyungsik Kim[a,*], Y. S. Kong[b], M. K. Yum[c] and J. T. Kim[d]

[a]*Department of Physics, Pukyong National University,*
Pusan 608-737, Korea
[b]*School of Ocean Engineering, Pukyong National University,*
Pusan 608-737, Korea
[c]*Department of Pediatric Cardiology, Hanyang University,*
Kuri 471 − 701, Korea
[d]*Department of Photonic Engineering, Chosun University,*
Gwangju 501 − 759, Korea

Abstract

The dynamical behavior for a quantum Brownian particle is investigated under a random potential of the fractional iterative map on a one-dimensional lattice. For our case, the quantum expectation values can be obtained numerically from the wave function of the fractional Schrödinger equation. Particularly, the square of mean displacement which is ensemble-averaged over our configuration is found to be proportional approximately to t^δ in the long time limit, where $\delta = 0.96 \pm 0.02$. The power-law behavior with scaling exponents $\epsilon = 0.98 \pm 0.02$ and $\theta = 0.51 \pm 0.01$ is estimated for $\overline{\langle p(t) \rangle^2}$ and $\overline{\langle f(t) \rangle^2}$, and the result presented is compared with other numerical calculations.

*Corresponding author. Tel.: +82-51-620-6354; fax: +82-51-611-6357.

E − mail address: kskim@pknu.ac.kr (Kyungsik Kim).

Until now, The Langevin equation has been played a important role in stochastic process, classically and quantum-mechanically. The classical Brownian motion has mainly been represented from the Langevin equation among the stochastic process problems [1]. The solution of this motion has analytically been obtained under the influence of a time-dependent fluctuation and known to be different from that of other motions which the stable probability distribution has usually the well-known Gaussian form. Furthermore, the natural generalization of the Brownian motion and the Wiener stochastic process [2] has been founded on the theory of stable probability distributions developed by Lévy.

On the other hand, the random motion of the Brownian particle has generally been described as the fluctuation-dissipation theorem [3]. In particular, the generalized Brownian motion [4, 5] has been showed to be equal to the Lioville equation that is presented from the dynamical variables in the phase space. The generalized Brownian motion has essentially been a determined equation which has more informations for the time-dependence of many-body system. As a counterpart, the quantum Brownian motion [6, 7] has been defined as the dynamics of a quantum particle under the the time-dependent potential and treated extensively with the transport of quantum excitation, the directed polymer problem, and quantum tunneling phenomena [8 − 10].

Recently, there have been investigated many diffusion processes deterministically by periodic iterated maps on the basis of the theory of dynamical systems. The nature of these diffusion motions is charaterized by the temporal scaling of the mean-square displacement $\sigma^2(t) \sim t^\nu$. For the normal diffusion $\nu = 1$, whereas in the subdiffusion $\nu < 1$, and in the superdiffusion $\nu > 1$. The subdiffusion has been arised typically in amorphous semiconductor [11] and polymer networks [12] and in porous media and fractal lattices [13]. The superdiffusion has been observed in rotating laminar fluid flows [14] and in enhanced transport of particles [15]. In particular, for the Kim-Kong map, the mean-square displacement for the resultant diffusive motion is found to scale approximately linearly with time for typical two control parameters [16].

In reality, the dynamical behavior of a quantum particle for the quantum Brownian motion has analytically and numerically been discussed from the square of the mean displacement [9, 10, 17] that is followed the scaling behavior of the width of the wave function, due to the unitary condition of the Schrödinger equation. To our knowledge, it is of special interest to be dealt with the random potential described by the form of fractional iterative map for the dynamical behavior of the quantum particle. Our extended random potential studied in this paper have not been fully explored up to now. Our purpose in this study is to present the dynamical behavior of a quantum particle through the fractional Schrödinger equation, where our potential is a time-dependent potential with the form of the fractional iterative map.

First of all the fractional Schrödinger equation is represented in terms of

$$i\frac{\partial^{\tau}}{\partial t^{\tau}}\Phi(x,t) = -\bigtriangledown^{2\alpha}\Phi(x,t) + W\rho(x,t)\Phi(x,t) \tag{1}$$

where $\Phi(x,t)$ is the wave function at displacment $x = x$ at time $t = t$. Eq. (1) is the fractional Schrödinger equation reduced by dimensionless quantities such that $m = \frac{1}{2}$ and $\frac{\hbar}{2m} = 1$. The space scaling value τ and the time scaling value α are, repectively, the fractional values in the range $0 < \tau \leq 1$ and $\frac{1}{2} < \alpha \leq 1$, W is the magnitute of a random potential, and $\rho(x,t)$ is a random function. Let's introduce that a random function given by the form of the fractional iterative map [18] at each displacement is represented as follows:

$$\rho(t) = \begin{cases} \gamma\exp[-\beta(-\log\rho(t-1))^{\eta}][1-\exp[-\beta(-\log\rho(t-1))^{\eta}]] & \text{for } 0 < \rho(t) \leq 1 \\ -\gamma\exp[-\beta(-\log\rho(t-1))^{\eta}][1-\exp[-\beta(-\log\rho(t-1))^{\eta}]] & \text{for } -1 \leq \rho(t) \leq 0, \end{cases} \tag{2}$$

where β and γ are control parameters of the fractional iterative map, and the ensemble-averaged values of a random function are

$$\overline{\rho(x,t)} = 0 \tag{3}$$

and

$$\overline{\rho(x,t)\ \rho(x',t')} = \rho^{2}(t)\ \delta_{xx'}\ \delta(t-t'). \tag{4}$$

3

We assume the initial condition that the quantum particle is distributed by a Gaussian packet centered at displacment $x = 0$ at time $t = 0$ with a width $\sigma = 10$:

$$\Phi(x,t) = \frac{1}{\sqrt{2\pi\sigma^2}} \, e^{-x^2/2\sigma^2}. \tag{5}$$

Since the derivative of order 2α for the wave function $\Phi(x,t)$ calculated by Crank-Nicolson method [19] can be written in terms of

$$\bigtriangledown^{2\alpha}\Phi(x,t) = \frac{\Phi(x+1,t) - 2\Phi(x,t) + \Phi(x-1,t)}{\Delta x^{2\alpha}}, \tag{6}$$

Eq.(1) can be derived as

$$
\begin{aligned}
\Phi(x,t+1) - \Phi(x,t) &= \frac{i\Delta t^\tau}{2}\{[\bigtriangledown^{2\alpha}\Phi(x,t+1) \\
&\quad + \bigtriangledown^{2\alpha}\Phi(x,t)] - \frac{W}{2}[\rho(x,t+1) \\
&\quad + \rho(x,t)][\Phi(x,t+1) + \Phi(x,t)]\}.
\end{aligned}
\tag{7}
$$

Hence the quantum expectation values [20] are defined by

$$\langle x(t) \rangle = \langle \Phi(x,t)|x|\Phi(x,t)\rangle, \tag{8}$$

$$\langle p(t) \rangle = -\frac{i}{2}\langle \Phi(x,t)|[x,H]|\Phi(x,t)\rangle, \tag{9}$$

and

$$\langle f(t) \rangle = -\frac{W}{2}\langle \Phi(x,t)|\rho(x+1,t) - \rho(x-1,t)|\Phi(x,t)\rangle. \tag{10}$$

Here $[x,H]$ denotes the commutator of position operator x and Hamiltonian operator H. In our scheme, we will calculate numerically three quantum expectation values $\overline{\langle x(t)\rangle^2}$, $\overline{\langle p(t)\rangle^2}$, and $\overline{\langle f(t)\rangle^2}$ from Eq.$(8) - (10)$, and our numerical result that we have obtained will be also compared with that of other random potential.

For the sake of simplicity, we conveniently restrict ourselves to the random function, i.e. the form of fractional iterative map, for three cases with $\eta = 1.6$, 1.8, and 2.0. In Eq. (2), we focus on a random function of the fractional iterative map with control parameters $\beta = 0.2$ and $\gamma = 3.78$. After finding the wave function $\Phi(x,t)$ in the special case of $\tau = 1$ and

$\alpha = 1$, we discuss numerically the diffusive dynamical behavior of the quantum particle from quantum expectation values. The quantum expectation values $\overline{\langle x(t) \rangle^2}$, $\overline{\langle p(t) \rangle^2}$, and $\overline{\langle f(t) \rangle^2}$ ensemble-averaged over 2×10^3 configurations are exactly found for fixed fractional value η and the magnitude of a random potential $W = 2, 5, 10$, and 15, and the result of these calculations is summarized in Table 1. Furthermore, Fig. 1 depicts $\overline{\langle x(t) \rangle^2}$ as a function of time t, while the plots of $\overline{\langle p(t) \rangle^2}$ and $\overline{\langle f(t) \rangle^2}$ are shown in Figs. 2 and 3, respectively. In the long time limit, it is obtained from Table 1 that $\overline{\langle x(t) \rangle^2} \propto t^\delta$ with $\delta = 0.96 \pm 0.02$, and that $\overline{\langle p(t) \rangle^2} \propto t^{-\epsilon}$ and $\overline{\langle f(t) \rangle^2} \propto t^{-\theta}$, where $\epsilon = 0.98 \pm 0.02$ and $\theta = 0.51 \pm 0.01$.

In conclusion, we have investigated the dynamical behavior for a fractional quantum Brownian particle under a random potential of the fractional iterative map on a one-dimensional lattice, and the scaling exponents for the quantum expectation value obtained have also discussed at three fractional values η. It has until now been well-known that $\overline{\langle x(t) \rangle^2} \propto t$ [10], if both the momentum and the random potential are given by the white noise. Moreover, since the expectation value $\overline{\langle x(t) \rangle^2}$ is proportional to t^3 for both the force and the random potential described by the white noise, this case has been known to be similar to that of a classical random walker with no dissipation [21]. In the large time limit, it has been reported [6] that the scaling exponents are $\delta = 0.5 \sim 0.6$ and $\epsilon = \theta = 0.5$ if a random potential has the pattern of the white noise. We obtain that our scaling exponents δ and ϵ take larger value than those of other result for a quantum particle characterized by the subdiffusive motion [9, 17]. Our case would be considered to be consistent with the normal diffusion because of the feature of chaotic orbital density of the fractional iterative map. It is also important to notice that our result for $\overline{\langle f(t) \rangle^2}$ is in agreement with previous estimates [6]. However, both cases can be recognized as resulting from the continuous spread of the quantum wave packet. In future, we will be treated extensively with the dynamical behavior for a fractional quantum Brownian particle under a random potential of the fractional iterative map, from the fractional Schrödinger equation at arbitrary fractional values τ and α. It is hoped that further detailed investigation for the fractional quantum Brownian motion will present ana-

lytically and numerically elsewhere.

References

[1] P. Langevin, Comptes Rend Acad. Sci. Paris **146**, 530 (1908); A. Einstein, *Theory of the Brownian Motion* (Dover, 1956); E. Nelson, *Dynamical Theories of Brownian Motion* (Princeton Univ. Press, 1967).

[2] C. W. Gardner, *Handbook of Stochastic Methods*, 2nd ed. (Springer-Verlag, Berlin, 1985).

[3] R. Kubo, Rep. Prog. Phys. **29**, 255 (1966).

[4] H. Mori, Prog. Theor. Phys. **33**, 423 (1965); **34**, 399 (1965).

[5] M. H. Lee, Phys. Rev. B**26**, 2547 (1982); Phys. Rev. Lett. **49**, 1072 (1982).

[6] P. Sheng and Z. Q. Zhang, Phys. Rev. B**48**, 609 (1993); A. A. Ovchinnikov and N. S. Erikhman, Sov. Phys. JETP **40**, 733 (1975).

[7] A. Madhukar and W. Post, Phys. Rev. Lett. **39**, 1424 (1977).

[8] S. Feng, L. Glolubovic and H. Sphohn, M. J. Feigenbaum, Phys. Rev. Lett. **65**, 1028 (1990).

[9] E. Medina, M. Kardar and H. Sphohn, Phys. Rev. Lett. **66**, 2176 (1991).

[10] L. Sand and M. Kardar, Phys. Rev. A**45**, 8859 (1992).

[11] H. Scher and E. W. Montroll, Phys. Rev. B **12**, 2455 (1975).

[12] F. Amblard, A. C. Maggs, B. Yurke, A. Pargellis and S. Leibler, Phys. Rev. Lett. **77**, 4470 (1996).

[13] S. Havlin and D. Ben-Avraham, Adv. in Phys. **36**, 695 (1987).

[14] T. H. Solomon, E. R. Weeks and H. Swinney, Phys. Rev. Lett. **71**, 3975 (1993).

[15] M. F. Schlesinger, B. J. West and J. Klafter, Phys. Rev. Lett. **58**, 1100 (1987).

[16] B. I. Henry, M. K. Yum, Y. S. Kong, J. S. Choi and Kyungsik Kim, Chaos, Solitons and Fractals (to be published).

[17] J. P. Bouchard, D. Touati and D. Sornette, Phy. Rev. Lett. **16**, 2115 (1991).

[18] Kyungsik Kim, G. H. Kim, J. R. Lee, J. S. Choi, Y. S. Kong, B. I. Henry, M. K. Yum and T. Odagaki, Fractals(to be published).

[19] G. D. Smith, *Numerical Solution of Partial Differential Equation* (Clarendon, Oxford, 1985)

[20] L. I. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw-Hill, New York, 1968)

[21] G. E. Uhlenbeck and L. S. Ornstein, Phy. Rev. **36**, 823 (1930).

FIGURE CAPTIONS

Figure 1. Plot of the quantum expectation value $\overline{\langle x(t) \rangle^2}$ as a function of the time t for a random function with $\eta = 1.6$ for $W = 2$, 5, 10, and 15, where $\delta = 0.94 \sim 0.98$.

Figure 2. Plot of the quantum expectation value $\overline{\langle p(t) \rangle^2}$ as a function of the time t for a random function with $\eta = 1.8$ for $W = 2$, 5, 10, and 15. The four lines have a near unity slope of $\epsilon = 0.96 \sim 1.00$ which is consistent with the normal diffusion.

Figure 3. Plot of the quantum expectation value $\overline{\langle f(t) \rangle^2}$ as a function of the time t for a random function with $\eta = 2.0$ for $W = 2$, 5, 10, and 15, where $\theta = 0.49 \sim 0.54$.

TABLE CAPTIONS

Table 1. Summary of values of the scaling exponents δ, ϵ, and θ in the case of three fractional values η.

	$W = 2$	$W = 5$	$W = 10$	$W = 15$
δ ($\eta = 1.6$)	0.96	0.98	0.94	0.94
ϵ ($\eta = 1.6$)	1.00	1.00	0.94	0.97
θ ($\eta = 1.6$)	0.50	0.52	0.51	0.52
δ ($\eta = 1.8$)	0.97	1.01	0.95	0.91
ϵ ($\eta = 1.8$)	1.00	0.96	0.96	0.96
θ ($\eta = 1.8$)	0.50	0.51	0.51	0.50
δ ($\eta = 2.0$)	0.95	0.97	1.00	1.00
ϵ ($\eta = 2.0$)	1.00	0.98	0.96	1.00
θ ($\eta = 2.0$)	0.49	0.51	0.53	0.54